The Lagrangian description of perfect fluids and modified gravity with an extra force

Valerio Faraoni*

Physics Department, Bishop's University
Sherbrooke, Québec, Canada J1M 1Z7

We revisit the issue of the correct Lagrangian description of a perfect fluid ($\mathcal{L}_1 = P$ versus $\mathcal{L}_2 = -\rho$) in relation with modified gravity theories in which galactic luminous matter couples nonminimally to the Ricci scalar. These Lagrangians are only equivalent when the fluid couples minimally to gravity and not otherwise; in the presence of nonminimal coupling they give rise to two distinct theories with different predictions.

INTRODUCTION

Attempts to explain the present acceleration of the universe discovered with type Ia supernovae [1] without invoking a mysterious and exotic dark energy [2] have lead to modifying gravity at the largest scales in the so-called $f(R)$ gravity ([3], see [4] for a comprehensive review and [5] for brief introductions). $f(R)$ theories of gravity have been employed also in attempts to explain dark matter in galaxies and clusters [6]. A recent proposal ny Bertolami, Böhmer, Harko, and Lobo (BBHL) [7] contemplates the possibility of coupling matter explicitly to the Ricci curvature, as described by the action [19]

$$S_{BBHL} = \int d^4x \sqrt{-g} \left\{ f_1(R) + [1 + \lambda f_2(R)] \mathcal{L}^{(m)} \right\},$$

(1)

where R is the Ricci curvature of spacetime, $f_{1,2}(R)$ are functions of R, λ is a small parameter, and $\mathcal{L}^{(m)}$ is the matter Lagrangian density (see [9] for similar theories and [10] for the special case of a gauge field coupling explicitly to the curvature). The extra coupling leads to the non-conservation of the matter stress-energy tensor $T_{ab}^{(m)}$, according to

$$\nabla^b T_{ab}^{(m)} = \frac{\lambda f_2'(R)}{1 + \lambda f_2(R)} \left[g_{ab}\mathcal{L}^{(m)} - T_{ab}^{(m)} \right] \nabla^b R,$$

(2)

where a prime denotes differentiation with respect to R. In [7] BBHL found that an extra force, not appearing in the minimally coupled versions of $f(R)$ gravity, acts upon particles and could effectively replace dark matter. This intriguing possibility has motivated further work [11, 12, 13].

In [7] a dust fluid model is assumed to describe ordinary (luminous) matter in galaxies and the extra force

$$f^a = \left(-\frac{\lambda f_2'}{1 + \lambda f_2} \nabla_b R + \frac{\nabla_b P}{P + \rho} \right) h^{ab}$$

(3)

was derived (here u^c is the fluid four-velocity and $h_{ab} \equiv g_{ab} + u_a u_b$). In [14] it was noted that, adopting the standard Lagrangian density $\mathcal{L}_1 = P$ for a perfect fluid, this extra force vanishes for dust, which has equation of state

$P = 0$ adequately reproducing the non-relativistic motions of stars, making again dark matter a necessary ingredient to explain the galactic rotation curves. Subsequently Bertolami and collaborators pointed out that an equivalent Lagrangian density for a perfect fluid is $\mathcal{L}_2 = -\rho$, which is obtained by adding surface terms to the action $S_1 = \int d^4x \sqrt{-g}\,\mathcal{L}_1$ [12] (see [15] for detailed studies of the Lagrangian formalism for perfect fluids [20]). From [12] it would appear that the extra force that has the potential to replace dark matter is present or absent according to the choice that is made between the two supposedly equivalent Lagrangians. This shows that, clearly, the two Lagrangian densities $\mathcal{L}_1 = P$ and $\mathcal{L}_2 = -\rho$ cannot be equivalent, and the recent literature has discussed this issue in terms of the problem of "which Lagrangian density correctly describes a perfect fluid". Here we show that, posed in these terms, this problem is meaningless. In fact, for a perfect fluid that does not couple explicitly to the curvature (*i.e.*, for $\lambda = 0$), the two Lagrangian densities $\mathcal{L}_1 = P$ and $\mathcal{L}_2 = -\rho$ are perfectly equivalent, as discussed extensively in Refs. [15] and remarked in [12]. However, there is little doubt that for a coupled perfect fluid ($\lambda \neq 0$) the two Lagrangians are inequivalent. This fact is a manifestation of a more general situation: if a Lagrangian system consists of two subsystems and there are two Lagrangians for one of these subsystems, which provide equivalent descriptions for that subsystems *when it is isolated*, the two Lagrangians cease to be equivalent *when the subsystem couples to the rest of the system*. An elementary example is provided in the next section. We follow the notations of Ref. [16].

TWO COUPLED OSCILLATORS

Point particle mechanics provides a very simple example of this situation. Considers two coupled one-dimensional oscillators which, in isolation, are described

by the Lagrangians

$$L_1\left(q_1, \dot{q}_1\right) = \frac{1}{2}\left(\dot{q}_1\right)^2 - \frac{K_1}{2}q_1^2 , \qquad (4)$$

$$L_2\left(q_2, \dot{q}_2\right) = \frac{1}{2}\left(\dot{q}_2\right)^2 - \frac{K_2}{2}q_2^2 , \qquad (5)$$

where $q_{1,2}$ are the respective Lagrangian coordinates. Now couple the two oscillators via the term $L_{12} = -\lambda q_1 L_2$ in the total Lagrangian

$$L\left(q_1, \dot{q}_1, q_2, \dot{q}_2\right) = L_1 + L_2 + L_{12} = \frac{1}{2}\left[\left(\dot{q}_1\right)^2 + \left(\dot{q}_2\right)^2\right]$$

$$-\frac{K_1}{2}q_1^2 - \frac{K_2}{2}q_2^2 - \lambda q_1\left[\frac{1}{2}\left(\dot{q}_2\right)^2 - \frac{K_2}{2}q_2^2\right] . \qquad (6)$$

The Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad (i = 1, 2) \qquad (7)$$

yield the equations of motion

$$\ddot{q}_1 + K_1 q_1 + \frac{\lambda}{2}\left[\left(\dot{q}_2\right)^2 - K_2 q_2^2\right] = 0 , \qquad (8)$$

$$\ddot{q}_2\left(1 - \lambda q_1\right) - \lambda \dot{q}_1 \dot{q}_2 + K_2\left(1 - \lambda q_1\right)q_2 = 0 . \qquad (9)$$

Now consider the new Lagrangian $L_2 + C$ (where C is a nonzero constant) for the subsystem 2: when this subsystem is isolated, this new Lagrangian is trivially equivalent to L_2. However, see what happens when L_2 is replaced by $L_2 + C$ in the description of the *coupled* subsystem; the equations of motion become then

$$\ddot{q}_1 + K_1 q_1 + \frac{\lambda}{2}\left[\left(\dot{q}_2\right)^2 - K_2 q_2^2\right] + \lambda C = 0 , \quad (10)$$

$$\ddot{q}_2\left(1 - \lambda q_1\right) - \lambda \dot{q}_1 \dot{q}_2 + K_2\left(1 - \lambda q_1\right)q_2 = 0 . \quad (11)$$

Eq. (10) does not coincide with (8).

To give another, less trivial, example assume that L_2 is replaced by the equivalent Lagrangian (when the subsystem 2 is isolated) $L_2 + dF/dt$, where $F(q_2, t)$ is a function of the coordinate q_2 and the time t. Then, the total Lagrangian of the two coupled subsytems becomes

$$L^* = L_1 + L_2 + \frac{dF}{dt} - \lambda q_1 L_2 - \lambda q_1 \frac{dF}{dt} ; \qquad (12)$$

the change, which leaves the equation of motion for the oscillator 2 unaffected when the latter is isolated, now affects the way this oscillator interacts with the first one. The new equations of motion are

$$\ddot{q}_1 + K_1 q_1 + \frac{\lambda}{2}\left[\left(\dot{q}_2\right)^2 - K_2 q_2^2\right] + \lambda \frac{\partial F}{\partial q_2}\dot{q}_2 + \lambda \frac{\partial F}{\partial t} = 0 ,$$
$$(13)$$

$$\ddot{q}_2\left(1 - \lambda q_1\right) - \lambda \dot{q}_1\left(\dot{q}_2 + \frac{\partial F}{\partial q_2}\right) + K_2\left(1 - \lambda q_1\right)q_2 = 0 .$$
$$(14)$$

DISCUSSION AND CONCLUSIONS

Another argument advocated in [12] as evidence for the extra force in galaxies needs to be addressed: in Ref. [17], Puetzfeld and Obukhov derive the equations of motion of a test body described by a Lagrangian density $\mathcal{L}$ and an energy-momentum tensor $T_{ab}^{(m)}$ in the modified gravity theories introduced in [7], using a multipole method. They show that, in general, extra forces occur even on single-pole particles, as described by eq. (32) of [17],

$$\frac{D}{D\tau}\left[mu^i\left(1 + \lambda f_2\right)\right] = N^{ib}\lambda\nabla_b f_2 . \qquad (15)$$

Here τ is the proper time along the worldline of a particle with four-tangent u^a and mass m, while

$$N^{ab} \equiv u^0 \Xi^{ab} = u^0 \int_{\Sigma(t)} d^3\vec{x}\,\tilde{\Xi}^{ab} , \qquad (16)$$

where a tilde denotes the density of the corresponding quantity and $\Xi^{ab} \equiv \mathcal{L}g^{ab}$ [17]. In spite of being a significant piece of work describing the motion of extended objects, the analysis of [17] does not solve the issue of which Lagrangian ($\mathcal{L}_1$ or $\mathcal{L}_2$) is appropriate to describe a perfect fluid: it merely says what are the equations of motion once a Lagrangian is assumed. If we assume the Lagrangian $\mathcal{L}_1 = P$, then $\Xi^{ab} = Pg^{ab}$ and, consequently, N^{ab}, vanish for a dust with $P = 0$. If instead the Lagrangian $\mathcal{L}_2 = -\rho$ is assumed, N^{ab} does not vanish and there are extra forces even on the particles of a dust fluid. Hence, the work of [17] des not resolve the issue of whether $\mathcal{L}_1$ or $\mathcal{L}_2$ is appropriate to describe a perfect fluid nonminimally coupled to gravity.

We note that the case of the cosmological constant, regarded as an effective form of matter, is very special: if one considers a cosmological constant as a perfect fluid with Lagrangian $\mathcal{L} = -\Lambda$, there is no difference between the two Lagrangians $\mathcal{L}_1 = P$ and $\mathcal{L}_2 = -\rho$ because of the peculiar equation of state $P = -\rho$ of the effective fluid, and there is no extra force on the cosmological constant "fluid". This coincidence is, of course, consistent with the fact that one can also regard the cosmological constant term as a purely geometrical term which is part of the first function $f_1(R)$ in the action (1) (for example, $f_1(R) = R - \Lambda$).

To summarize our conclusions, there is little doubt that the two Lagrangian densities $\mathcal{L}_1 = P$ and $\mathcal{L}_2 = -\rho$ are equivalent for the description of a perfect fluid which is not coupled directly to gravity, as shown in Refs. [15]. However, as soon as this fluid is coupled explicitly to gravity as in eq. (1), the two Lagrangian densities cease to be equivalent. It is not clear that one should be physically preferred with respect to the other: simply, they give rise to two inequivalent theories of gravity and matter, which are both correct. Which one should be chosen must be decided by independent arguments. It is a fact

that by choosing $\mathcal{L} = P$ there is no extra force on a dust fluid and it is equally undeniable that by choosing $\mathcal{L}_2 = -\rho$ there will be such a force, which may ultimately provide an alternative to dark matter. We are not able to provide independent arguments in favour of one choice or the other: the contribution of the present note merely consists in showing that, in spite of the appearance, there is no contradiction between the results of [14] and those of [12].

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* vfaraoni@ubishops.ca

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[19] This action was introduced in [8].

[20] The Lagrangian density $\mathcal{L}_2 = -\rho$ is used less frequently than $\mathcal{L}_1 = P$; it appears, *e.g.*, in Ref. [18].